

ON FUNCTIONS GIVEN BY ALGEBRAIC POWER SERIES OVER HENSELIAN VALUED FIELDS

KRZYSZTOF JAN NOWAK

ABSTRACT. This paper provides, over Henselian valued fields, some theorems on implicit function and of Artin–Mazur on algebraic power series. Also discussed are certain versions of the theorems of Abhyankar–Jung and Newton–Puiseux. The latter is used in analysis of functions of one variable, definable in the language of Denef–Pas, to obtain a theorem on existence of the limit, proven over rank one valued fields in one of our recent papers. This result along with the technique of fiber shrinking (developed there over rank one valued fields) were, in turn, two basic tools in the proof of the closedness theorem.

1. INTRODUCTION

Let K be a Henselian valued field of equicharacteristic zero with valuation v , valuation ring R and value group Γ . The main purpose of this paper is to examine algebraic power series over K and continuous functions given by them. To this end, in Section 2 we state a version of the implicit function theorem, and in the next section prove one of the Artin–Mazur theorem on algebraic power series. Consequently, every algebraic power series over K determines a unique continuous function which is definable in the language of valued fields. Section 4 presents certain versions of the theorems of Abhyankar–Jung and Newton–Puiseux for Henselian subalgebras of formal power series which are closed under power substitution and division by a coordinate, given in our paper [9].

The last section is devoted to functions of one variable definable in the language of Denef–Pas. Having the Newton–Puiseux theorem for algebraic power series at hand, we briefly outline how to adapt the proof of the theorem on existence of the limit which was proven over rank one valued fields in our paper [10, Proposition 5.2]. This result

2000 *Mathematics Subject Classification.* 12J25, 13B40, 14P20.

Key words and phrases. Implicit function, density property, the theorems of Artin–Mazur, Abhyankar–Jung and Newton–Puiseux, definable functions of one variable.

along with the technique of fiber shrinking from [10, Section 6] were, in turn, two basic tools used in the proof of the closedness theorem [10, Theorem 3.1] over Henselian rank one valued fields.

We should finally mention that the closedness theorem enables i.a. application of resolution of singularities and transformation to a normal crossing by blowing up in much the same way as over the locally compact ground field. Some other its applications in Henselian geometry are provided in our recent papers [11, 12].

2. SOME VERSIONS OF THE IMPLICIT FUNCTION THEOREM

In this section, we give elementary proofs of some versions of the inverse mapping and implicit function theorems (cf. the versions established in the papers [13, Theorem 7.4] or [7, Proposition 3.1.4]). We begin with a simplest version (H) of Hensel's lemma in several variables, studied by Fisher [6]. Given an ideal \mathfrak{m} of a ring R , let $\mathfrak{m}^{\times n}$ stand for the n -fold Cartesian product of \mathfrak{m} and R^\times for the set of units of R . The origin $(0, \dots, 0) \in R^n$ is denoted by $\mathbf{0}$.

(H) *Assume that a ring R satisfies Hensel's conditions (i.e. it is linearly topologized, Hausdorff and complete) and that an ideal \mathfrak{m} of R is closed. Let $f = (f_1, \dots, f_n)$ be an n -tuple of restricted power series $f_1, \dots, f_n \in R\{X\}$, $X = (X_1, \dots, X_n)$, J be its Jacobian determinant and $a \in R^n$. If $f(\mathbf{0}) \in \mathfrak{m}^{\times n}$ and $J(\mathbf{0}) \in R^\times$, then there is a unique $a \in \mathfrak{m}^{\times n}$ such that $f(a) = \mathbf{0}$.*

Proposition 2.1. *Under the above assumptions, f induces a bijection*

$$\mathfrak{m}^{\times n} \ni x \longrightarrow f(x) \in \mathfrak{m}^{\times n}$$

of $\mathfrak{m}^{\times n}$ onto itself.

Proof. For any $y \in \mathfrak{m}^{\times n}$, apply condition (H) to the restricted power series $f(X) - y$. \square

If, moreover, the pair (R, \mathfrak{m}) satisfies Hensel's conditions (i.e. every element of \mathfrak{m} is topologically nilpotent), then condition (H) holds by [1, Chap. III, §4.5].

Remark 2.2. Henselian local rings can be characterized both by the classical Hensel lemma and by condition (H): a local ring (R, \mathfrak{m}) is Henselian iff (R, \mathfrak{m}) with the discrete topology satisfies condition (H) (cf. [6, Proposition 2]).

Now consider a Henselian local ring (R, \mathfrak{m}) . Let $f = (f_1, \dots, f_n)$ be an n -tuple of polynomials $f_1, \dots, f_n \in R[X]$, $X = (X_1, \dots, X_n)$ and J be its Jacobian determinant.

Corollary 2.3. *Suppose that $f(\mathbf{0}) \in \mathfrak{m}^{\times n}$ and $J(\mathbf{0}) \in R^\times$. Then f is a homeomorphism of $\mathfrak{m}^{\times n}$ onto itself in the \mathfrak{m} -adic topology. If, in addition, R is a Henselian valued ring with maximal ideal \mathfrak{m} , then f is a homeomorphism of $\mathfrak{m}^{\times n}$ onto itself in the valuation topology.*

Proof. Obviously, $J(a) \in R^\times$ for every $a \in \mathfrak{m}^{\times n}$. Let \mathcal{M} be the jacobian matrix of f . Then

$$f(a + x) - f(a) = \mathcal{M}(a) \cdot x + g(x) = \mathcal{M}(a) \cdot (x + \mathcal{M}(a)^{-1} \cdot g(x))$$

for an n -tuple $g = (g_1, \dots, g_n)$ of polynomials $g_1, \dots, g_n \in (X)^2 R[X]$. Hence the assertion follows easily. \square

The proposition below is a version of the inverse mapping theorem.

Proposition 2.4. *If $f(\mathbf{0}) = \mathbf{0}$ and $e := J(\mathbf{0}) \neq 0$, then f is an open embedding of $e^2 \cdot \mathfrak{m}^{\times n}$ into $e \cdot \mathfrak{m}^{\times n}$.*

Proof. Let \mathcal{N} be the adjugate of the matrix $\mathcal{M}(\mathbf{0})$ and $y = e^2 b$ with $b \in \mathfrak{m}^{\times n}$. Since

$$f(eX) = e \cdot \mathcal{M}(\mathbf{0}) \cdot X + e^2 g(X)$$

for an n -tuple $g = (g_1, \dots, g_n)$ of polynomials $g_1, \dots, g_n \in (X)^2 R[X]$, we get the equivalences

$$f(eX) = y \Leftrightarrow f(eX) - y = \mathbf{0} \Leftrightarrow e \cdot \mathcal{M}(\mathbf{0}) \cdot (X + \mathcal{N}g(X) - \mathcal{N}b) = \mathbf{0}.$$

Applying Corollary 2.3 to the map $h(X) := X + \mathcal{N}g(X)$, we get

$$f^{-1}(y) = ex \Leftrightarrow x = h^{-1}(\mathcal{N}b) \text{ and } f^{-1}(y) = eh^{-1}(\mathcal{N} \cdot y/e^2).$$

This finishes the proof. \square

Further, let $0 \leq r < n$, $p = (p_{r+1}, \dots, p_n)$ be an $(n - r)$ -tuple of polynomials $p_{r+1}, \dots, p_n \in R[X]$, $X = (X_1, \dots, X_n)$, and

$$J := \frac{\partial(p_{r+1}, \dots, p_n)}{\partial(X_{r+1}, \dots, X_n)}, \quad e := J(\mathbf{0}).$$

Suppose that

$$\mathbf{0} \in V := \{x \in R^n : p_{r+1}(x) = \dots = p_n(x) = 0\}.$$

In a similar fashion as above, we can establish the following version of the implicit function theorem.

Proposition 2.5. *If $e \neq 0$, then there exists a continuous map*

$$\phi : (e^2 \cdot \mathfrak{m})^{\times r} \longrightarrow (e \cdot \mathfrak{m})^{\times (n-r)}$$

which is definable in the language of valued fields and such that $\phi(0) = 0$ and the graph map

$$(e^2 \cdot \mathfrak{m})^{\times r} \ni u \longrightarrow (u, \phi(u)) \in (e^2 \cdot \mathfrak{m})^{\times r} \times (e \cdot \mathfrak{m})^{\times(n-r)}$$

is an open embedding into the zero locus V of the polynomials p .

Proof. Put $f(X) := (X_1, \dots, X_r, p(X))$; of course, the jacobian determinant of f at $\mathbf{0} \in R^n$ is equal to e . Keep the notation from the proof of Proposition 2.4, take any $b \in e^2 \cdot \mathfrak{m}^{\times r}$ and put $y := (e^2 b, 0) \in R^n$. Then we have the equivalences

$$f(eX) = y \Leftrightarrow f(eX) - y = \mathbf{0} \Leftrightarrow e\mathcal{M}(\mathbf{0}) \cdot (X + \mathcal{N}g(X) - \mathcal{N} \cdot (b, 0)) = \mathbf{0}.$$

Applying Corollary 2.3 to the map $h(X) := X + \mathcal{N}g(X)$, we get

$$f^{-1}(y) = ex \Leftrightarrow x = h^{-1}(\mathcal{N} \cdot (b, 0)) \quad \text{and} \quad f^{-1}(y) = eh^{-1}(\mathcal{N} \cdot y/e^2).$$

Therefore the function

$$\phi(u) := eh^{-1}(\mathcal{N} \cdot (u, 0)/e^2)$$

is the one we are looking for. \square

3. DENSITY PROPERTY AND A VERSION OF THE ARTIN–MAZUR THEOREM OVER HENSELIAN VALUED FIELDS

We say that a topological field K satisfies the *density property* (cf. [8, 10]) if the following equivalent conditions hold.

- (1) If X is a smooth, irreducible K -variety and $\emptyset \neq U \subset X$ is a Zariski open subset, then $U(K)$ is dense in $X(K)$ in the K -topology.
- (2) If C is a smooth, irreducible K -curve and $\emptyset \neq U$ is a Zariski open subset, then $U(K)$ is dense in $C(K)$ in the K -topology.
- (3) If C is a smooth, irreducible K -curve, then $C(K)$ has no isolated points.

(This property is indispensable for ensuring reasonable topological and geometric properties of algebraic subsets of K^n ; see [10] for the case where the ground field K is a Henselian rank one valued field.) The density property of Henselian non-trivially valued fields follows immediately from Proposition 2.5 and the Jacobian criterion for smoothness (see e.g. [4, Theorem 16.19]), recalled below for the reader's convenience.

Theorem 3.1. *Let $I = (p_1, \dots, p_s) \subset K[X]$, $X = (X_1, \dots, X_n)$ be an ideal, $A := K[X]/I$ and $V := \text{Spec}(A)$. Suppose the origin $\mathbf{0} \in K^n$*

lies in V (equivalently, $I \subset (X)K[X]$) and V is of dimension r at $\mathbf{0}$. Then the Jacobian matrix

$$\mathcal{M} := \left[\frac{\partial p_i}{\partial X_j}(\mathbf{0}) : i = 1, \dots, s, j = 1, \dots, n \right]$$

has rank $\leq (n-r)$ and V is smooth at $\mathbf{0}$ iff \mathcal{M} has exactly rank $(n-r)$. Furthermore, if V is smooth at $\mathbf{0}$ and

$$\mathcal{J} := \frac{\partial(p_{r+1}, \dots, p_n)}{\partial(X_{r+1}, \dots, X_n)}(\mathbf{0}) = \det \left[\frac{\partial p_i}{\partial X_j}(\mathbf{0}) : i, j = r+1, \dots, n \right] \neq 0,$$

then p_{r+1}, \dots, p_n generate the localization $I \cdot K[X]_{(X_1, \dots, X_n)}$ of the ideal I with respect to the maximal ideal (X_1, \dots, X_n) .

Remark 3.2. Under the above assumptions, consider the completion

$$\widehat{A} = K[[X]]/I \cdot K[[X]]$$

of A in the (X) -adic topology. If $\mathcal{J} \neq 0$, it follows from the implicit function theorem for formal power series that there are unique power series

$$\phi_{r+1}, \dots, \phi_n \in (X_1, \dots, X_r) \cdot K[[X_1, \dots, X_r]]$$

such that

$$p_i(X_1, \dots, X_r, \phi_{r+1}(X_1, \dots, X_r), \dots, \phi_n(X_1, \dots, X_r)) = 0$$

for $i = r+1, \dots, n$. Therefore the homomorphism

$$\widehat{\alpha} : \widehat{A} \longrightarrow K[[X_1, \dots, X_r]], \quad X_j \mapsto X_j, \quad X_k \mapsto \phi_k(X_1, \dots, X_r),$$

for $j = 1, \dots, r$ and $k = r+1, \dots, n$, is an isomorphism.

Conversely, suppose that $\widehat{\alpha}$ is an isomorphism; this means that the projection from V onto $\text{Spec } K[X_1, \dots, X_r]$ is etale at $\mathbf{0}$. Then the local rings A and \widehat{A} are regular and, moreover, it is easy to check that the determinant $\mathcal{J} \neq 0$ does not vanish after perhaps renumbering the polynomials $p_i(X)$.

We say that a formal power series $\phi \in K[[X]]$, $X = (X_1, \dots, X_n)$, is algebraic if it is algebraic over $K[X]$. The kernel of the homomorphism of K -algebras

$$\sigma : K[X, T] \longrightarrow K[[X]], \quad X_1 \mapsto X_1, \dots, X_n \mapsto X_n, \quad T \mapsto \phi(X),$$

is, of course, a principal prime ideal:

$$\ker \sigma = (p) \subset K[X, T],$$

where $p \in K[X, T]$ is a unique (up to a constant factor) irreducible polynomial, called an *irreducible polynomial* of ϕ .

We now state a version of the Artin–Mazur theorem (cf. [2, 3] for the classical versions).

Proposition 3.3. *Let $\phi \in (X)K[[X]]$ be an algebraic formal power series. Then there exist polynomials*

$$p_1, \dots, p_r \in K[X, Y], \quad Y = (Y_1, \dots, Y_r),$$

and formal power series $\phi_2, \dots, \phi_r \in K[[X]]$ such that

$$\mathcal{J} := \frac{\partial(p_1, \dots, p_r)}{\partial(Y_1, \dots, Y_r)}(\mathbf{0}) = \det \left[\frac{\partial p_i}{\partial Y_j}(\mathbf{0}) : i, j = 1, \dots, r \right] \neq 0,$$

and

$$p_i(X_1, \dots, X_n, \phi_1(X), \dots, \phi_r(X)) = 0, \quad i = 1, \dots, r,$$

where $\phi_1 := \phi$.

Proof. Let $p_1(X, Y_1)$ be an irreducible polynomial of ϕ_1 . Then the integral closure B of $A := K[X, Y_1]/(p_1)$ is a finite A -module and thus is of the form

$$B = K[X, Y]/(p_1, \dots, p_s), \quad Y = (Y_1, \dots, Y_r),$$

where $p_1, \dots, p_s \in K[X, Y]$. Obviously, A and B are of dimension n , and the induced embedding $\alpha : A \rightarrow K[[X]]$ extends to an embedding $\beta : B \rightarrow K[[X]]$. Put

$$\phi_k := \beta(Y_k) \in K[[X]], \quad k = 1, \dots, r.$$

Substituting $Y_k - \phi_k(0)$ for Y_k , we may assume that $\phi_k(0) = 0$ for all $k = 1, \dots, r$. Hence $p_i(\mathbf{0}) = 0$ for all $i = 1, \dots, s$.

The completion \widehat{B} of B in the (X, Y) -adic topology is a local ring of dimension n , and the induced homomorphism

$$\widehat{\beta} : \widehat{B} = K[[X, Y]]/(p_1, \dots, p_s) \longrightarrow K[[X]]$$

is, of course, surjective. But, by the Zariski main theorem (cf. [14, Chap. VIII, § 13, Theorem 32]), \widehat{B} is a normal domain. Comparison of dimensions shows that $\widehat{\beta}$ is an isomorphism. Now, it follows from Remark 3.2 that the determinant $\mathcal{J} \neq 0$ does not vanish after perhaps renumbering the polynomials $p_i(X)$. This finishes the proof. \square

Propositions 3.3 and 2.5 immediately yield the following

Corollary 3.4. *Let $\phi \in (X)K[[X]]$ be an algebraic power series with irreducible polynomial $p(X, T) \in K[X, T]$. Then there is an $a \in K$, $a \neq 0$, and a unique continuous function*

$$\widetilde{\phi} : a \cdot R^n \longrightarrow K$$

which is definable in the language of valued fields and such that $\tilde{\phi}(0) = 0$ and $p(x, \tilde{\phi}(x)) = 0$ for all $x \in a \cdot R^n$. \square

For simplicity, we shall denote the induced continuous function by the same letter ϕ . This abuse of notation will not lead to confusion in general.

Remark 3.5. Clearly, the ring $K[[X]]_{alg}$ of algebraic power series is the henselization of the local ring $K[X]_{(X)}$ of regular functions. Therefore the implicit functions $\phi_{r+1}(u), \dots, \phi_n(u)$ from Proposition 2.5 correspond to unique algebraic power series

$$\phi_{r+1}(X_1, \dots, X_r), \dots, \phi_n(X_1, \dots, X_r)$$

without constant term. In fact, one can deduce by means of the classical version of the implicit function theorem for restricted power series (cf. [1, Chap. III, §4.5] or [6]) that $\phi_{r+1}, \dots, \phi_n$ are of the form

$$\phi_k(X_1, \dots, X_r) = e \cdot \omega_k(X_1/e^2, \dots, X_r/e^2), \quad k = r+1, \dots, n,$$

where $\omega_k(X_1, \dots, X_r) \in R[[X_1, \dots, X_r]]$ and $e \in R$.

4. THE NEWTON–PUISEUX AND ABHYANKAR–JUNG THEOREMS

Here we are going to provide a version of the Newton–Puiseux theorem, which will be used in analysis of definable functions of one variable in the next section.

We call a polynomial

$$f(X; T) = T^s + a_{s-1}(X)T^{s-1} + \dots + a_0(X) \in K[[X]][T],$$

$X = (X_1, \dots, X_s)$, quasiordinary if its discriminant $D(X)$ is a normal crossing:

$$D(X) = X^\alpha \cdot u(X) \quad \text{with} \quad \alpha \in \mathbb{N}^s, \quad u(X) \in k[[X]], \quad u(0) \neq 0.$$

Let K be an algebraically closed field of characteristic zero. Consider a henselian $K[X]$ -subalgebra $K\langle X \rangle$ of the formal power series ring $K[[X]]$ which is closed under reciprocal (whence it is a local ring), power substitution and division by a coordinate. For positive integers r_1, \dots, r_n put

$$K\langle X_1^{1/r_1}, \dots, X_n^{1/r_n} \rangle := \{a(X_1^{1/r_1}, \dots, X_n^{1/r_n}) : a(X) \in K\langle X \rangle\};$$

when $r_1 = \dots = r_m = r$, we denote the above algebra by $K\langle X^{1/r} \rangle$.

In our paper [9], we established a version of the Abhyankar–Jung theorem.

Proposition 4.1. *Under the above assumptions, every quasiordinary polynomial*

$$f(X; T) = T^s + a_{s-1}(X)T^{s-1} + \cdots + a_0(X) \in K\langle X \rangle[T]$$

has all its roots in $K\langle X^{1/r} \rangle$ for some $r \in \mathbb{N}$; actually, one can take $r = s!$.

A particular case is the following version of the Newton-Puiseux theorem.

Corollary 4.2. *Let X denote one variable. Every polynomial*

$$f(X; T) = T^s + a_{s-1}(X)T^{s-1} + \cdots + a_0(X) \in K\langle X \rangle[T]$$

has all its roots in $K\langle X^{1/r} \rangle$ for some $r \in \mathbb{N}$; one can take $r = s!$. Equivalently, the polynomial $f(X^r, T)$ splits into T -linear factors. If $f(X, T)$ is irreducible, then $r = s$ will do and

$$f(X^s, T) = \prod_{i=1}^s (T - \phi(\epsilon^i X)),$$

where $\phi(X) \in K\langle X \rangle$ and ϵ is a primitive root of unity.

Remark 4.3. Since the proof of these theorems is of finitary character, it is easy to check that if the ground field K of characteristic zero is not algebraically closed, they remain valid for the Henselian subalgebra $\overline{K} \otimes_K K\langle X \rangle$ of $\overline{K}[[X]]$, where \overline{K} denotes the algebraic closure of K .

The ring $K[[X]]_{alg}$ of algebraic power series is a local Henselian ring closed under power substitutions and division by a coordinate. Thus the above results apply to the algebra $K\langle X \rangle = K[[X]]_{alg}$.

5. DEFINABLE FUNCTIONS OF ONE VARIABLE

At this stage, we can readily proceed with analysis of definable functions of one variable over arbitrary Henselian valued fields of equicharacteristic zero. We wish to obtain a general version of the theorem on existence of the limit stated below. It was proven in [10, Proposition 5.2] over rank one valued fields. Now the language \mathcal{L} under consideration is the three sorted language of Denef–Pas.

Proposition 5.1. *(Existence of the limit) Let $f : A \rightarrow K$ be an \mathcal{L} -definable function on a subset A of K and suppose 0 is an accumulation point of A . Then there is a finite partition of A into \mathcal{L} -definable sets A_1, \dots, A_r and points $w_1, \dots, w_r \in \mathbb{P}^1(K)$ such that*

$$\lim_{x \rightarrow 0} f|_{A_j}(x) = w_j \quad \text{for } j = 1, \dots, r.$$

Moreover, there is a neighbourhood U of 0 such that each definable set $\{(v(x), v(f(x))) : x \in (A_j \cap U) \setminus \{0\}\} \subset \Gamma \times (\Gamma \cup \{\infty\})$, $j = 1, \dots, r$, is contained in an affine line with rational slope

$$l = \frac{p_j}{q} \cdot k + \beta_j, \quad j = 1, \dots, r,$$

with $p_j, q \in \mathbb{Z}$, $q > 0$, $\beta_j \in \Gamma$, or in $\Gamma \times \{\infty\}$. \square

Proof. Having the Newton–Puiseux theorem for algebraic power series at hand, we can repeat mutatis mutandis the proof from loc. cit. as briefly outlined below. In that paper, the field L is the completion of the algebraic closure \overline{K} of the ground field K . Here, in view of Corollary 4.3, the K -algebras $L\{X\}$ and $\widehat{K}\{X\}$ should be just replaced with $\overline{K} \otimes_K K[[X]]_{\text{alg}}$ and $K[[X]]_{\text{alg}}$, respectively. Then the reasonings follow almost verbatim. Note also that Lemma 5.1 (to the effect that K is a closed subspace of \overline{K}) holds true for arbitrary Henselian valued fields. \square

We conclude with the following comment. The above proposition along with the technique of fiber shrinking from [10, Section 6] were two basic tools in the proof of the closedness theorem [10, Theorem 3.1] over Henselian rank one valued fields, which plays an important role in Henselian geometry.

REFERENCES

- [1] N. Bourbaki, *Algèbre Commutative*, Hermann, Paris, 1962.
- [2] M. Artin, B. Mazur, *On periodic points*, Ann. Math. **81** (1965), 82–99.
- [3] J. Bochnak, M. Coste, M.-F. Roy, *Real Algebraic Geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 36, Springer-Verlag, Berlin, 1998.
- [4] D. Eisenbud, *Commutative Algebra with a View Towards Algebraic Geometry*, Graduate Texts in Math. **150**, Springer-Verlag, New York, 1994.
- [5] A.J. Engler, A. Prestel, *Valued Fields*, Springer-Verlag, Berlin, 2005.
- [6] B. Fisher, *A note on Hensel’s lemma in several variables*, Proc. Amer. Math. Soc. **125** (11) (1997), 3185–3189.
- [7] O. Gabber, P. Gille, L. Moret-Bailly, *Fibrés principaux sur les corps valués henséliens*, Algebraic Geometry **1** (2014), 573–612.
- [8] J. Kollár, K. Nowak, *Continuous rational functions on real and p -adic varieties*, Math. Zeitschrift **279** (2015), 85–97.
- [9] K.J. Nowak, *Supplement to the paper “Quasianalytic perturbation of multiparameter hyperbolic polynomials and symmetric matrices”* (Ann. Polon. Math. **101** (2011), 275–291), Ann. Polon. Math. **103** (2012), 101–107.
- [10] K.J. Nowak, *Some results of algebraic geometry over Henselian rank one valued fields*, Sel. Math. New Ser. **23** (2017), 455–495.
- [11] K.J. Nowak, *Hölder and Lipschitz continuity of functions definable over Henselian rank one valued fields*, arXiv:1702.03463 [math.AG].

- [12] K.J. Nowak, *Piecewise continuity of functions definable over Henselian rank one valued fields*, arXiv:1702.07849 [math.AG].
- [13] A. Prestel, M. Ziegler, *Model theoretic methods in the theory of topological fields*, J. Reine Angew. Math. **299–300** (1978), 318–341.
- [14] O. Zariski, P. Samuel, *Commutative Algebra*, Vol. II, Van Nostrand, Princeton, 1960.

Institute of Mathematics
Faculty of Mathematics and Computer Science
Jagiellonian University
ul. Profesora Łojasiewicza 6, 30-348 Kraków, Poland
E-mail address: `nowak@im.uj.edu.pl`